Rutgers University: Algebra Written Qualifying Exam

August 2018: Problem 1 Solution

Exercise. Give an example of an integral domain R and an ideal I in R such that *all* of the following statements hold. The ideal I is not principal, it is not maximal, and it is prime.

Solution.

$$\boxed{\langle 2, x\rangle \subseteq \mathbb{Z}[x,y] = R}$$

Not Principal: If $\langle 2, x\rangle = \langle p(x,y)\rangle$ then $2 \in I \implies p(x,y)q(x,y) = 2$ for some $q(x,y) \in \mathbb{Z}[x,y]$
$\implies p(x,y) = 1$ or 2
If $p = 1$ then $\langle 2, x\rangle = \mathbb{Z}[x,y]$, which is not true, so we have a contradiction.
If $p = 2$ then $x \in \langle 2\rangle$ and so $\exists f(x,q) \in \mathbb{Z}[x,y]$ such that $2f(x,y) = x$, which is not true, so we have a contradiction.
$\langle 2, x\rangle$ is not a principal ideal in $R = \mathbb{Z}[x,y]$.

Not Maximal: $\langle 2, x\rangle\langle 2, x, y\rangle \subsetneq R$

Prime: Let $a(x,y), b(x,y) \in R$. Then

$$a(x,y) = a'(x,y) + r(y), \qquad b(x,y) = b'(x,y) + s(y)$$

where $a'(x,y), b'(x,y) \in I$ and $r(y) = \sum \alpha_i y^i$ and $s(y) = \sum \beta_i y^i$ where α_i and β_i are either 0 or odd integers
Suppose $ab \in I$. Then

$$a(x,y)b(x,y) = \underbrace{a'(x,y)b'(x,y) + a'(x,y)s(y) + b'(x,y)r(y)}_{\in I} + r(y)s(y)$$

and $r(y)s(y) = \sum \alpha_i\beta_j y^{i+j}$ and α_i, β_j are either zero or odd integers
$\implies \alpha_i\beta_j$ is either 0 or an odd integer
$\alpha_i\beta_j \in I \iff r(y)s(y) = 0$
If $r(y)s(y) = 0$, then $r(y) = 0$ or $s(y) = 0$
$\implies a \in I$ or $b \in I$.
$\langle 2, x\rangle$ is a prime ideal.